NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

British Mathematical Olympiad

21st March, 1978

Time allowed - $3\frac{1}{2}$ hours

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order.

Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

In any question, marks may be added for elegance and clarity or subtracted for obscure or poor presentation.

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1. Determine with proof the point P inside a given triangle ABC for which the product PL.PM.PN is a maximum, where L,M,N are the feet of the perpendiculars from P to BC,CA,AB respectively.

2. Prove that there is no proper fraction $\frac{m}{n}$, with denominator $n \leqslant 100$, whose decimal expansion contains the block of consecutive digits 167 in that order.

3. Show that there is one and only one sequence $\{u_n\}$ of integers such that $u_1 = 1$, $u_1 < u_2$, and

   $$u_n^3 + 1 = u_{n-1}u_{n+1} \text{ for all } n > 1.$$

4. An altitude of a tetrahedron is a line through a vertex perpendicular to the opposite face.

   Prove that the four altitudes of a tetrahedron are concurrent if and only if each edge of the tetrahedron is perpendicular to its opposite edge.

5. Inside a cube of side 15 units there are 11000 given points.

   Prove that there is a sphere of unit radius within which there are at least 6 of the given points.

6. Show that if n is a non-zero integer, $2\cos n\theta$ is a polynomial of the nth degree in $2\cos\theta$.

   Hence or otherwise prove that if k is rational then $\cos k\pi$ either is equal to one of the numbers $0, \pm\frac{1}{2}, \pm 1$, or is irrational.